

DISCRETE MIXED VOLUME AND HODGE-DELIGNE NUMBERS

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ABSTRACT. Generalizing the famous Bernstein-Kushnirenko theorem, Khovanskii proved in 1978 a combinatorial formula for the arithmetic genus of the compactification of a generic complete intersection associated to a family of lattice polytopes. Recently, an analogous combinatorial formula, called the discrete mixed volume, was introduced by Bihan and shown to be nonnegative. By making a footnote of Khovanskii in his paper explicit, we interpret this invariant as the (motivic) arithmetic genus of the non-compact generic complete intersection associated to the family of lattice polytopes.

INTRODUCTION

The by now classical Bernstein-Kushnirenko theorem (also called BKK theorem) is a gem linking solutions of systems of polynomial equations to the combinatorics of lattice polytopes [Ber75]. It states that the number of common solutions in $(\mathbb{C}^*)^n$ to n general equations is given by the mixed volume of the n associated Newton polytopes. The latter can be computed as the alternating sum of the number of lattice points in Minkowski sums of these n lattice polytopes.

Recently, there has been significant progress on the combinatorial side in the case of k lattice polytopes $P_1, \dots, P_k \subset \mathbb{R}^n$ where $k < n$. For a set of indices $I \subseteq [k] := \{1, \dots, k\}$ we write P_I for the Minkowski sum

$$P_I := \sum_{i \in I} P_i,$$

and $P_\emptyset := \{0\}$. Bihan [Bih16] calls the alternating sum

$$\text{DMV}(P_1, \dots, P_k) := \sum_{I \subseteq [k]} (-1)^{k-|I|} |P_I \cap \mathbb{Z}^n|$$

the *discrete mixed volume* of P_1, \dots, P_k and proves that it is always non-negative. We remark that for $k = n$ this is precisely the mixed volume. More generally, the polynomial

$$\text{ME}(P_1, \dots, P_k; m) := \text{DMV}(mP_1, \dots, mP_k)$$

is called the *mixed Ehrhart polynomial* [HJKST15]. We recall that the *Ehrhart polynomial* of a lattice polytope $P \subset \mathbb{R}^n$ is given by $\text{ehr}(P)(m) := |mP \cap \mathbb{Z}^n|$ for $m \in \mathbb{N}$. New work by Jochemko and Sanyal generalizes Bihan's positivity from counting lattice points to more general valuations [JS16].

In this note, we clarify the algebraic geometric implications of these combinatorial non-negativity results. We verify in Theorem 1.6 that the discrete mixed volume $\text{ME}(P_1, \dots, P_k; 1)$ equals what we call, in the spirit of [Yok11], the *motivic arithmetic genus* of a general complete intersection in $(\mathbb{C}^*)^n$ corresponding to P_1, \dots, P_k . This statement was already hinted at in a footnote of Khovanskii [Kho78, p. 41]. It is natural to ask, more generally, whether or not the motivic arithmetic genus is non-negative for every smooth subvariety of the torus.

For smooth projective varieties, this motivic arithmetic genus specializes to the usual arithmetic genus (see Remark 1.4). In 1978 Khovanskii [Kho78, Theorem 1] proved that in our setting the arithmetic genus of a compactified smooth complete intersection equals $\text{ME}(P_1, \dots, P_k; -1)$. In combination, these two results may be seen as a motivic reciprocity theorem.

This note is organized as follows. In Section 1 we recall the Hodge-Deligne polynomial, define the motivic arithmetic genus, and state our main result (Theorem 1.6). Its proof is given in Section 2.

1. GENUS FORMULAE FOR COMPLETE INTERSECTIONS

In this section, we set the notation, state our main result, and present some open questions.

1.1. Hodge-Deligne polynomials and the motivic arithmetic genus. Let us recall definition and properties of Hodge-Deligne polynomials. We refer to [DK86] for more details.

Given a quasi-projective variety Y over \mathbb{C} , the cohomology with compact supports $H_c^k(Y, \mathbb{Q}) \otimes \mathbb{C}$ carries a natural mixed Hodge structure [Del71]. The dimension of the (p, q) -piece is denoted by $h^{(p,q)}(H_c^k(Y))$ giving rise to the (p, q) Euler characteristic

$$e^{(p,q)}(Y) := \sum_k (-1)^k h^{(p,q)}(H_c^k(Y)).$$

If Y is smooth and projective, the Hodge structure is pure so that we only have one summand, $e^{(p,q)}(Y) = (-1)^{p+q} h^{(p,q)}(Y)$, the usual Hodge number.

The generating function for these numbers

$$E(Y; u, v) := \sum_{p,q} e^{p,q}(Y) u^p v^q \in \mathbb{Z}[u, v]$$

is the *Hodge-Deligne polynomial* (or *E-polynomial*).

All we need to know about these polynomial invariants is that they behave nicely under stratifications.

Theorem 1.1. *The invariant E factors through the Grothendieck ring, i.e.,*

- (1) $E(\{\text{point}\}) = 1$,
- (2) if $X = X_1 \sqcup X_2$ with $X_i \subset X$ locally closed for $i = 1, 2$, then $E(X) = E(X_1) + E(X_2)$, and
- (3) $E(X_1 \times X_2) = E(X_1) \cdot E(X_2)$.

It follows that E behaves multiplicatively on fibrations.

Corollary 1.2. *If $\pi: Y \rightarrow X$ is Zariski-locally trivial with fiber F , then $E(Y) = E(X) \cdot E(F)$.*

Example 1.3. Using these tools, we can compute the E -polynomial for toric varieties $X(\Sigma)$ from the f -vector of the defining fan Σ . As \mathbb{P}^1 is smooth and projective with Betti numbers $(1, 0, 1)$, we must have $E(\mathbb{P}^1) = uv + 1$ whence $E(\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}) = E(\mathbb{P}^1) - 2 = uv - 1$ so that $E((\mathbb{C}^*)^d) = (uv - 1)^d$. Using the stratification of $X(\Sigma)$ by tori, we get

$$E(X(\Sigma)) = \sum_d f_d(\Sigma) (uv - 1)^{n-d}.$$

We define

$$e^{p,+}(Y) := \sum_q e^{p,q}(Y) \in \mathbb{Z}$$

the p^{th} χ_y -characteristic. In particular, $e^{0,+}(Y) = E(Y; 0, 1)$. We denote

$$(-1)^{\dim(Y)} E(Y; 0, 1)$$

as the *motivic arithmetic genus* of a (not necessarily compact) variety Y .

Remark 1.4. In the traditional situation of a nonsingular projective variety X , we use the term *arithmetic genus* instead of motivic arithmetic genus. We remark that Khovanskii uses in [Kho78] the term ‘arithmetic genus’ for $E(X; 0, 1)$ while it refers to $(-1)^{\dim(X)}(E(X; 0, 1) - 1)$ in Hartshorne [Har77, III, Ex.5.3]. We prefer the above definition as it will fit nicely to the combinatorial notion. Observe that by using the birational invariance of the arithmetic genus, Khovanskii defines in [Kho78] even the arithmetic genus of non-compact varieties as the arithmetic genus of some/any smooth projective compactification.

1.2. Our setup. Let $P_1, \dots, P_k \subset \mathbb{R}^n$ be lattice polytopes, where we do not impose any additional restrictions on their dimensions. Let $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$ be a generic k -tuple of Laurent polynomials with Newton polytopes P_1, \dots, P_k , respectively. Then we denote by Y the associated complete intersection in $(\mathbb{C}^*)^n$ of dimension $n - r$ defined by $f_1 = \dots = f_k = 0$. We choose a compactification \bar{Y} in a nonsingular projective toric variety such that \bar{Y} is nonsingular and intersects all torus orbits transversally.

In the simplest example $k = 1, n = 2$, of a single polynomial in two variables, \bar{Y} will be a smooth curve of genus $g = |\text{relint}(P) \cap \mathbb{Z}^2|$ so that $E(\bar{Y}) = uv - g(u + v) + 1$ while Y has $b = |\partial P \cap \mathbb{Z}^2|$ points removed, so that $E(Y) = uv - g(u + v) - (b - 1)$.

1.3. The main result. Let us recall Khovanskii's formula for the arithmetic genus of \bar{Y} . Here, $\text{relint}(P)$ denotes the relative interior of a polytope P . Recall that we agreed on $P_\emptyset = \{0\}$ so that $|\text{relint}(P_\emptyset) \cap \mathbb{Z}^n| = 1$.

Theorem 1.5 (Khovanskii [Kho78]). *In the notation of Subsection 1.2, for $Y \neq \emptyset$,*

$$(-1)^{\dim(Y)} E(\bar{Y}; 0, 1) = \sum_{I \subseteq [k]} (-1)^{\dim(P_I) - |I|} |\text{relint}(P_I) \cap \mathbb{Z}^n|.$$

Ehrhart-Macdonald reciprocity implies $\text{ehr}(P_I)(-1) = (-1)^{\dim(P_I)} |\text{relint}(P_I) \cap \mathbb{Z}^n|$, see [BR07]. Hence,

$$(-1)^{\dim(Y)} E(\bar{Y}; 0, 1) = \text{ME}(P_1, \dots, P_k; -1).$$

We remark that in contrast to the geometric genus the arithmetic genus is not necessarily nonnegative. For instance, choose P_1 with vertices $(0, 0, 0), (a, 0, 0), (0, a, 0)$, and P_2 with vertices $(0, 0, 1), (1, 0, 1)$. Then $\text{ME}(P_1, P_2; -1) < 0$ for $a \gg 0$.

As Khovanskii indicated in the footnote on p.41 of [Kho78], there is a corresponding result in the non-compact situation. Here, for $\beta \in \mathbb{Z}_{\geq 0}^I$, we define $|\beta| := \sum_{i \in I} \beta_i$. Moreover, we set $\mathbb{Z}^\emptyset := \{0\}$.

Theorem 1.6. *In the notation of Subsection 1.2,*

$$e^{p,+}(Y) = (-1)^{n-p} \sum_{I \subseteq [k]} (-1)^{|I|} \left(\sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^I \\ |\beta| \leq p}} (-1)^{|\beta|} \binom{n + |I|}{p - |\beta|} |(P_I + P_\beta) \cap \mathbb{Z}^n| \right)$$

In particular, we get for $p = 0$

$$e^{0,+}(Y) = \sum_{I \subseteq [k]} (-1)^{n-|I|} |P_I \cap \mathbb{Z}^n| = (-1)^{n-k} \text{DMV}(P_1, \dots, P_k)$$

In other words,

$$(-1)^{\dim(Y)} E(Y; 0, 1) = \text{DMV}(P_1, \dots, P_k) = \text{ME}(P_1, \dots, P_k; 1)$$

Corollary 1.7. *The motivic arithmetic genus of a generic complete intersection in the algebraic torus associated to a family of lattice polytopes is nonnegative. The generic complete intersection is non-empty if and only if the motivic arithmetic genus is positive.*

Proof. Nonnegativity of the discrete mixed volume is the central result in [Bih16]. It remains to prove the second statement. In Theorem 3.17 of [JS16] it is shown that the discrete mixed volume of P_1, \dots, P_n is positive if and only if there are linearly independent segments $S_1 \subseteq P_1, \dots, S_k \subseteq P_k$ with vertices in \mathbb{Z}^k . By the proof of Lemma 5.1.9 in [Sch14] this is equivalent to P_1, \dots, P_k satisfying the ‘1-independence’ condition in Definition 3.1 in [BB96b]. Theorem 3.3 in [BB96b] yields that this condition is equivalent to the nonemptiness of the complete intersection. \square

Our proof of Theorem 1.6 follows directly the ideas outlined in [DK86]. In this fundamental paper, a formula for the χ_y -characteristic was given in the case of Y being a hypersurface, and an algorithm on how to generalize from hypersurfaces to complete intersections was described. Let us also remark that a complete formula for the Hodge-Deligne polynomial of Y in the case of a hypersurface was given in [BB96a, Theorem 3.24].

2. PROOF OF THEOREM 1.6

2.1. Ehrhart theory. Given a lattice polytope $P \subset \mathbb{R}^n$, the *Ehrhart polynomial* of P is given by $\text{ehr}_P(k) := |kP \cap \mathbb{R}^n|$ for $k \in \mathbb{Z}_{\geq 0}$. The *Ehrhart generating function* is of the form

$$\sum_{j=0}^{\infty} \text{ehr}(P; j) t^j = \frac{\sum_{k=0}^n h_k^*(P)}{(1-t)^{\dim(P)+1}},$$

where $h_0^*(P), \dots, h_d^* \in \mathbb{Z}_{\geq 0}$, and $h_k^* = 0$ for $k > \dim(P)$. For $k = 0, \dots, n$, we have the following relation

$$(1) \quad h_k^*(P) = \sum_{j=0}^k (-1)^{k-j} \binom{\dim(P)+1}{k-j} \text{ehr}(P; j)$$

We say for lattice polytopes $P, Q \subset \mathbb{R}^n$ that P is a *lattice pyramid* over Q if there is an affine-linear transformation of \mathbb{R}^n bijectively mapping \mathbb{Z}^n onto \mathbb{Z}^n such that Q is mapped onto $Q' \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ and P is mapped onto the convex hull of Q' and $(0, \dots, 0, 1)$. The h^* -coefficients are invariant under lattice pyramid constructions.

Let P_1, \dots, P_k be given as in Subsection 1.2. For $I \subseteq [k]$ we define the *Cayley polytope*

$$C_I = \text{Cayley}(P_i : i \in I)$$

as the lattice polytope in \mathbb{R}^{n+k} with vertices $P_i \times \{e_i\}$ for $i \in I$ (and $C_\emptyset := \emptyset$). Then C_I is a lattice polytope of dimension $\dim C_I = \dim(P_I) + |I| - 1$. For $\alpha \in \mathbb{Z}_{\geq 0}^k$ we define $|\alpha| := \alpha_1 + \dots + \alpha_k$ and $\text{supp}(\alpha) := \{i \in [k] : \alpha_i \neq 0\}$. We define the Minkowski sum

$$P_\alpha := \sum_{i \in I} \alpha_i P_i.$$

With this notation we can compute the Ehrhart polynomial of C_I as follows.

$$(2) \quad \text{ehr}(C_I; j) = \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^k \\ |\alpha| = j}} |P_\alpha \cap \mathbb{Z}^n| = \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^k \\ |\alpha| = j, \text{supp}(\alpha) \subseteq I}} |P_\alpha \cap \mathbb{Z}^n|$$

2.2. Some binomial identities. Let us recall the following binomial identities (e.g., see Table 169 in [GKP94]):

$$(3) \quad \sum_{s \in \mathbb{Z}} \binom{a}{q+s} \binom{b}{w+s} = \binom{a+b}{a-q+w}$$

for $a, b \in \mathbb{Z}_{\geq 0}$ and $q, w \in \mathbb{Z}$.

$$(4) \quad \sum_{s=0}^{\infty} (-1)^s \binom{a}{s} \binom{b+s}{q} = (-1)^a \binom{b}{q-a}$$

for $a \in \mathbb{Z}_{\geq 0}$ and $b, q \in \mathbb{Z}$.

2.3. A special case. The following result is the key situation in the proof of Theorem 1.6.

Lemma 2.1. *Let $P \subset \mathbb{R}^d$ be a lattice pyramid over a lattice polytope $\emptyset \neq Q \subset \mathbb{R}^d$. We denote by $Z \subset (\mathbb{C}^*)^d$ the generic hypersurface associated to P . Then*

$$e^{p,+}(Z) = (-1)^{d-1-p} \left(\binom{d}{p+1} + \sum_{j=0}^{\infty} (-1)^{j+1} \binom{d}{d-p+j-1} \text{ehr}(Q; j) \right).$$

Proof. Let $\dim(P) = d - c$, so $\dim(Q) = d - 1 - c$. As Z is generic, up to isomorphism we can assume that $Z = Z' \times (\mathbb{C}^*)^c$ for $Z' \subset (\mathbb{C}^*)^{d-c}$ generic hypersurface associated to $P \subset \mathbb{R}^{d-c}$. Therefore, $E(Z; u, v) = E(Z'; u, v)(uv - 1)^c$. This implies

$$(5) \quad e^{p,+}(Z) = \sum_{m \in \mathbb{Z}} e^{m,+}(Z')(-1)^{m+c-p} \binom{c}{p-m}$$

By [DK86, 4.6] we have for $m \geq 0$

$$(-1)^{d-c-1} e^{m,+}(Z') = (-1)^m \binom{d-c}{m+1} + h_{m+1}^*(P).$$

Note that this equation also holds for $m < 0$. Plugging this into (5) yields

$$\begin{aligned} e^{p,+}(Z) &= \sum_{m \in \mathbb{Z}} \left[(-1)^{d-c-1} \left((-1)^m \binom{d-c}{m+1} + h_{m+1}^*(P) \right) \right] (-1)^{m+c-p} \binom{c}{p-m} \\ &= (-1)^{d-1-p} \left[\left(\sum_{m \in \mathbb{Z}} \binom{d-c}{m+1} \binom{c}{c-p+m} \right) + \sum_{m \in \mathbb{Z}} (-1)^m \binom{c}{p-m} h_{m+1}^*(P) \right] \end{aligned}$$

Binomial identity (3) shows that the expression in the round parentheses evaluates to $\binom{d}{d-p-1} = \binom{d}{p+1}$. As $h_{m+1}^*(P) = h_{m+1}^*(Q)$, by equation (1) the previous expression equals

$$\begin{aligned} &(-1)^{d-1-p} \left(\binom{d}{p+1} + \sum_{m \in \mathbb{Z}} (-1)^m \binom{c}{p-m} \sum_{j=0}^{\infty} (-1)^{m+1-j} \binom{d-c}{m+1-j} \text{ehr}(Q; j) \right) \\ &= (-1)^{d-1-p} \left[\binom{d}{p+1} + \sum_{j=0}^{\infty} (-1)^{j+1} \text{ehr}(Q; j) \left(\sum_{m \in \mathbb{Z}} \binom{c}{c-p+m} \binom{d-c}{1-j+m} \right) \right] \end{aligned}$$

Binomial identity (3) shows that the sum in the round parentheses evaluates to $\binom{d}{p+1-j} = \binom{d}{d-p+j-1}$. This finishes the proof. \square

2.4. Proof of Theorem 1.6. We follow the procedure described in [DK86, 6.2] on how to reduce the computation of the Hodge-Deligne polynomials from the complete intersection case to that of a hypersurface (also called the Cayley trick). Let $\tilde{Z} \subset (\mathbb{C}^*)^n \times \mathbb{C}^k$ be the hypersurface of $F = 1 + \sum y_i f_i \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_k]$. As $\tilde{Z} \rightarrow (\mathbb{C}^*)^n \setminus Y$ is a bundle with fiber \mathbb{C}^{k-1} over $(\mathbb{C}^*)^n \setminus Y$, we get from Corollary 1.2

$$E(\tilde{Z}; u, v) = (uv)^{k-1} [(uv - 1)^n - E(Y; u, v)]$$

Therefore,

$$E(Y; u, v) = (uv - 1)^n - \frac{1}{(uv)^{k-1}} E(\tilde{Z}; u, v)$$

This yields,

$$e^{p,+}(Y) = (-1)^{n-p} \binom{n}{p} - \sum_{q=0}^{\infty} e^{p+k-1, q+k-1}(\tilde{Z}).$$

As $(uv)^{k-1} | E(\tilde{Z}; u, v)$, we have $e^{p+k-1, q'}(\tilde{Z}) = 0$ for $q' < k - 1$, thus

$$(6) \quad e^{p,+}(Y) = (-1)^{n-p} \binom{n}{p} - e^{p+k-1,+}(\tilde{Z}).$$

For $I \subseteq [k]$ let us define the strata

$$Z_I := \tilde{Z} \cap \{y_j \neq 0 : j \in I\} \cap \{y_j = 0 : j \notin I\}$$

Because $\tilde{Z} = \bigsqcup_I Z_I$ (and $Z_\emptyset = \emptyset$) we get

$$(7) \quad E(\tilde{Z}; u, v) = \sum_{\emptyset \neq I \subseteq [k]} E(Z_I; u, v).$$

For $I \neq \emptyset$, by construction, Z_I is the generic hypersurface in $(\mathbb{C}^*)^{n+|I|}$ associated to a lattice pyramid over the Cayley polytope $C_I \neq \emptyset$. Now, we are in the special case of Lemma 2.1. Hence, we get for $I \neq \emptyset$

$$e^{p+k-1,+}(Z_I) = (-1)^{n+|I|-p-k} \left(\binom{n+|I|}{p+k} + \sum_{j=0}^{\infty} (-1)^{j+1} \binom{n+|I|}{n+|I|-p-k+j} \text{ehr}(C_I; j) \right)$$

Therefore, from this equation together with (6) and (7), we see that

$$(8) \quad e^{p,+}(Y) = (-1)^{n-p} \binom{n}{p} + (-1)^{n+1-p-k} (\text{expression}_1 - \text{expression}_2),$$

where

- expression_1 equals

$$\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} \binom{n+|I|}{p+k}$$

- expression_2 equals

$$\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} \sum_{j=0}^{\infty} (-1)^j \binom{n+|I|}{p+k-j} \text{ehr}(C_I; j)$$

Here, expression_1 can be rewritten as

$$\left(\sum_{s=0}^{\infty} (-1)^s \binom{k}{s} \binom{n+s}{p+k} \right) - \binom{n}{p+k}$$

which gets simplified by binomial expression (4) to

$$(9) \quad (-1)^k \binom{n}{p} - \binom{n}{p+k}$$

It remains to consider expression_2 . By (2) it evaluates to

$$\begin{aligned} & \sum_{j=0}^{\infty} (-1)^j \sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} \binom{n+|I|}{p+k-j} \sum_{|\alpha|=j, \text{supp } \alpha \subseteq I} \text{ehr}(P_{\alpha}; 1) \\ &= \left(\sum_{\alpha \in \mathbb{Z}_{\geq 0}^k} \text{ehr}(P_{\alpha}; 1) (-1)^{|\alpha|} \sum_{[k] \supseteq I \supseteq \text{supp } \alpha} (-1)^{|I|} \binom{n+|I|}{p+k-|\alpha|} \right) - \binom{n}{p+k} \end{aligned}$$

Let us define $t_{\alpha} := |\text{supp } \alpha|$. By introducing the counting variable $s = |I \setminus \text{supp } \alpha|$, the previous expression becomes

$$\left(\sum_{\alpha \in \mathbb{Z}_{\geq 0}^k} \text{ehr}(P_{\alpha}; 1) (-1)^{|\alpha|+t_{\alpha}} \left[\sum_{s=0}^{k-t_{\alpha}} (-1)^s \binom{k-t_{\alpha}}{s} \binom{n+t_{\alpha}+s}{p+k-|\alpha|} \right] \right) - \binom{n}{p+k}$$

Binomial identity (4) shows that the sum in the brackets evaluates to $(-1)^{k-t_{\alpha}} \binom{n+t_{\alpha}}{p+t_{\alpha}-|\alpha|}$. Hence, we can rewrite the previous expression as

$$\left(\sum_{\alpha \in \mathbb{Z}_{\geq 0}^k} (-1)^{k+|\alpha|} \binom{n+t_{\alpha}}{p+t_{\alpha}-|\alpha|} \text{ehr}(P_{\alpha}; 1) \right) - \binom{n}{p+k}$$

Writing α as the characteristic vector of its support plus a vector β reformulates expression_2 as

$$(10) \quad \left(\sum_{I \subseteq [k]} (-1)^{k-|I|} \sum_{\beta \in \mathbb{Z}_{\geq 0}^I} (-1)^{|\beta|} \binom{n+|I|}{p-|\beta|} \text{ehr}(P_I + P_{\beta}; 1) \right) - \binom{n}{p+k}$$

Finally, plugging (9) and (10) into (8) and some simplification yields

$$e^{p,+}(Y) = (-1)^{n-p} \sum_{I \subseteq [k]} (-1)^{|I|} \sum_{\beta \in \mathbb{Z}_{\geq 0}^I} (-1)^{|\beta|} \binom{n+|I|}{p-|\beta|} \text{ehr}(P_I + P_\beta; 1)$$

□

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